

# Motional dressed states in a Bose condensate: Superfluidity at supersonic speed

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We present an exact analytic solution of a nonlinear Schrödinger field interacting with a moving potential (obstacle) at supersonic speed. We show that the field forms a stable shape-invariant structure localized around the obstacle — a dressing effect that protects the field against excitations by the obstacle.

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One of the most intriguing phenomena in superfluid is frictionless flow below a critical velocity  $v_c$  [1]. This phenomenon, first observed in liquid helium, demonstrates key features of collective behavior in macroscopic quantum coherent systems. Recently, experimental evidence of the existence of a critical velocity in a Bose-Einstein condensed gas has been found [2]. It is known that atoms in such a dilute system interact weakly with each other, and so it justifies the use of the nonlinear Schrödinger equation (NLSE) to study the quantum dynamics of the condensate. Indeed, direct numerical solutions of the NLSE have indicated a distinct transition from superfluid flow to resistive flow at a critical velocity [3,4]. Recent studies have also addressed dynamic features in solutions of the NLSE, such as phonon and vortex emissions [5–8], in order to understand the dissipation mechanisms.

As an obstacle moves through a Bose condensate, the regime of supersonic speed is often considered as a dissipation domain where frictionless motion disappears. This is understood from Landau's argument that energy-momentum conservation forbids phonon excitations unless the obstacle's velocity is at least the speed of sound  $c$ , i.e., the critical velocity  $v_c$  equals  $c$  for systems with a pure phonon spectrum [9]. In fact, for realistic systems obeying the NLSE,  $v_c$  is *less* than  $c$  because of the reduced fluid densities appearing at the boundary of the obstacle [3–5,10]. Therefore the speed of sound seems to be an upper limit beyond which dissipation occurs generally. However, we shall show that there exist interesting exceptions at least in one dimensional systems. Such exceptions exhibit rich dynamic features resulted from coherent field-obstacle interactions, which cannot be captured by the concept of critical velocities alone.

In this paper we present an exact analytic solution of a nonlinear Schrödinger field interacting with a moving repulsive potential (obstacle) in one dimension. Our solution represents a family of *motional dressed states* in which the field organizes itself as a stable and shape invariant structure localized at the moving obstacle. Once

the dressing is fully developed, there is no energy transfer from the obstacle to the field. We show that this happens when the velocity of the potential is *greater* than the speed of sound in the field. Therefore dressed states formation is a novel coherent feature that can be maintained in a condensate at high speeds. The existence of dressed states at supersonic speed requires a specific matching between the field and the potential. Our solution provides a prescription for the matching potentials. This opens possibilities of coherent control of local properties of a Bose condensate.

We begin by writing down a nonlinear Schrödinger equation for a field  $\Psi(x, t)$  under the influence of a moving potential  $U(x - vt)$ , which is well localized and repulsive. As usual, the NLSE in a dimensionless form is given by

$$i \frac{\partial}{\partial t} \Psi(x, t) = -\frac{1}{2} \frac{\partial^2}{\partial x^2} \Psi(x, t) + |\Psi(x, t)|^2 \Psi(x, t) + U(x - vt) \Psi(x, t) - \Psi(x, t). \quad (1)$$

This equation can be used to determine the evolution of the condensate wavefunction in one dimension at zero temperature. If the condensate has a chemical potential  $\mu$  and a particle mass  $m$ , then the position  $x$  and the time  $t$  are in units of  $\hbar/\sqrt{m\mu}$  and  $\hbar/\mu$ , respectively. In Eq. (1), the nonlinear term is positive for repulsive self interactions, and the speed  $v$  is taken to be positive for definiteness. We assume that the field is uniform (i.e., at rest) as  $x$  tends to  $\pm\infty$ . This imposes the boundary condition

$$\Psi(x \rightarrow \pm\infty, t) = e^{i\gamma_{\pm}} \quad (2)$$

at any finite time  $t$ . Here  $\gamma_{\pm}$  are constants which account for possibilities of having different phases at the two infinities. The choice of unit modulus is a convenient scale which gives the speed of sound  $c = 1$  for undisturbed systems.

We define a dressed state of the system by a solution of (1) in the shape invariant form:  $\Psi(x, t) = \psi(x - vt)$  subjected to the boundary condition (2). In other words dressed states are stationary states in the co-moving frame of the potential. For  $v$  in the subsonic domain  $v < 1$ , Hakim has reported stationary states which are of the form of a dip surrounding the obstacle [10]. The dip appearance is understood because of the repulsive character of the potential. However, for  $v$  greater than the speed of sound, no known steady states exist. We now describe a method to determine dressed states in this regime.

By assuming the solution in the form:  $\psi(\tau) = r(\tau)e^{i\theta(\tau)}$ , where  $\tau \equiv x - vt$ , Eq. (1) requires

$$\frac{1}{2}(\ddot{r} - r\dot{\theta}^2) = (r^3 - r) + U(\tau)r - vr\dot{\theta} \quad (3)$$

$$\frac{1}{2}(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = v\dot{r}. \quad (4)$$

Here the derivatives are taken with respect to  $\tau$ , i.e.,  $\dot{r} = dr/d\tau$ ,  $\ddot{r} = d^2r/d\tau^2$ , etc. If we consider  $\tau$  as a kind of time, then equations (3) and (4) are equivalent to the motion of a classical particle interacting with a central force and a time-dependent magnetic field described by  $U(\tau)$ . Therefore a dressed state corresponds to a trajectory matching the boundary conditions (2). This requires

$$r(\tau \rightarrow \pm\infty) = 1, \quad (5)$$

$$\theta(\tau \rightarrow \pm\infty) = \gamma_{\pm}. \quad (6)$$

From the constant of motion  $K \equiv vr^2 - r^2\dot{\theta}$ , we see that  $K$  must equal  $v$  for desired trajectories. In this way we have  $\dot{\theta} = v(1 - r^{-2})$ , and so Eqs. (3) and (4) are reduced to a one-dimensional trajectory problem [10],

$$\frac{1}{2}\ddot{r} = (r^3 - r) - \frac{v^2}{2}\left(r - \frac{1}{r^3}\right) + U(\tau)r. \quad (7)$$

Once the function  $U(\tau)$  is given, we may search for a solution by finding out initial conditions  $r(0)$  and  $\dot{r}(0)$  that make  $r(\tau)$  approach one asymptotically. Although the search can be performed systematically via numerical means, the existence of a solution is not a guarantee for arbitrary functions  $U$  and speeds  $v$ .

We obtain dressed state solutions analytically by adopting an inverse approach. First we construct an  $r(\tau)$  that satisfies the boundary condition (5), then we determine the corresponding  $U(\tau)$  from Eq. (7). The requirement is that the  $U(\tau)$  must be *well localized and everywhere repulsive*, i.e.,  $U'(\xi > 0) < 0$  and  $U'(\xi < 0) > 0$ . This approach tells us how one should design the shape of the  $U$  in order to have a specified form of  $\Psi$ . The matching of  $U(\tau)$  and  $r(\tau)$  yields the exact dressed state solutions. We note that in experimental situations [2],  $U$  can in fact be an adjustable function controlled by the intensity of a detuned laser.

We give a family of exact solutions in the form of a solitary wave:

$$r(\tau) = \left[1 - \frac{\alpha}{v}\text{sech}^2(\beta\tau)\right]^{-1/2} \quad (8)$$

$$\theta(\tau) = \frac{\alpha}{\beta}\tanh\beta\tau. \quad (9)$$

The two *positive* parameters  $\alpha$  and  $\beta$  characterize the height and inverse width of the field  $\Psi$  near the moving potential. The exact form of  $U(\tau)$  that matches Eq. (8) is given by,

$$U(\tau) = \frac{\alpha\text{sech}^2(\beta\tau)}{2[v - \alpha\text{sech}^2(\beta\tau)]^2} \{ \lambda_1 + \lambda_2\text{sech}^2(\beta\tau) + \lambda_3\text{sech}^4(\beta\tau) + \lambda_4\text{sech}^6(\beta\tau) \} \quad (10)$$

where  $\lambda_1 = 2v(v^2 + \beta^2 - 1)$ ,  $\lambda_2 = 2\alpha + \alpha\beta^2 - 3\beta^2v - 5\alpha v^2$ ,  $\lambda_3 = 4\alpha^2v$  and  $\lambda_4 = -\alpha^3$ . We find that if  $v$  satisfies

$$\frac{(v - \alpha)^3}{v} - 1 \geq 2\beta^2, \quad (11)$$

then  $U(\tau)$  in Eq. (10) is repulsive as required. Indeed,  $v$  has to be supersonic in order to satisfy the inequality (11). Therefore we have provided an explicit solution of dressed states at supersonic speeds.

In Fig. 1, we illustrate the general shapes of the matching potentials  $U(\tau)$  as a function of parameters  $\alpha$  and  $\beta$ . We plot a characteristic curve defined by the equality sign in (11). Therefore the region under the curve corresponds to purely repulsive potentials. These potentials have a maximum barrier height at  $\tau = 0$ . Above the curve, the second derivative  $U''(0)$  is positive which corresponds to an attractive force on particles near the center of  $U$ . We shall not discuss the solutions in such a domain, since the potentials are not purely repulsive. We remark that at a higher speed  $v$ , the characteristic curve takes a higher position on the  $\alpha - \beta$  plane, allowing a wider range of parameters for the solution (8-9). We also note that  $U(\tau)$  takes a simpler form,  $U(\tau) \approx v\alpha\text{sech}^2(\beta\tau)$ , when  $v \gg \alpha, \beta$ .

There are interesting features in our solution: (I) A dressed state for a *repulsive*  $U$  at  $v > 1$  must come with a *positive*  $\alpha$ . This means that the field concentrates at the center of  $U$ , even though  $U$  itself is repulsive. Such a ‘counter-intuitive’ behavior does not occur for dressed states in the subsonic speed domain. It is worth noting that the family of solutions Eqs. (8-10) can be generalized for subsonic speeds. In that case,  $\alpha$  is negative which recovers the expected dip at the center of the potential. (II) The height of the matching potential  $U$  is bounded by

$$2U(0) \leq (v^2 - 3v^{2/3} + 2). \quad (12)$$

This inequality sets an upper limit of  $U(0)$  for the existence of the solution (8-9). From the right side of Eq. (7), we see that a large  $U$  term would lead to an unbounded  $r(\tau)$  unless the  $v^2$  term can make a balance. Hence for speeds  $v$  which are close to (but still greater than) 1, the matching potential must be weaker accordingly. It is also apparent that there are no dressed states for impenetrable potentials.

In order to investigate the dynamical consequences of dressed states, we solve numerically the time-dependent NLSE (1) for  $\Psi(x, t)$  which starts from the ground state  $\Psi(x, 0) = 1$ . The potential  $U(x - vt)$  is turned on suddenly at  $t = 0$ . In Fig. 2a, we plot the field energy,  $\Delta E(t) = \frac{1}{2} \int dx (|\nabla\Psi(x, t)|^2 + |\Psi(x, t)|^4) - E(0)$  relative to the initial field energy  $E(0)$  as a function of time. This quantity tells us how much field energy is gained from the moving potential if it is switched off at a time  $t$ . In Fig. 2a we see that there is a positive gain of the field energy at early times. After a characteristic time of order  $2\beta^{-1}$ , the energy eventually becomes a constant. In

this steady state regime, the moving potential does not ‘feel’ any the reaction force from the field even though  $U$  is repulsive.

The emergence of a steady state indicates the formation of a dressed state. This is explicitly shown in Fig. 2b where  $|\Psi|$  are plotted as a function of  $x$  at different times. The arrows in the figure indicate the positions of the moving potential. We see that the field develops a shape invariant wavepacket form at the potential, i.e., the potential is dressed. By comparing both the phase and the amplitude of the wavepacket with the analytic expressions Eqs. (8) and (9), we found a very good agreement between them. At regions far away from the potential,  $\Psi$  exhibits features created by the sudden motion at  $t = 0$ . We see that the left-most wavefront propagates at the speed of sound, and there is an oscillatory pattern developing in a depression region near the middle. These features are partly responsible for the transient energy gain shown in Fig. 2a. A closer look at the depression region at longer times (not shown) indicates that the short wavelength structure becomes grey solitons traveling at subsonic speeds [11].

We emphasize that the usual picture of critical velocities does not apply to dressed states. The velocity-dependent requirements here are specified by the inequalities (11) and (12), which do not impose an upper limit for  $v$ . Instead,  $v$  has to be sufficiently high in order to achieve dressing for a given  $U$  [12]. We have calculated the energy gain using the *same* potential shown in Fig. 2a but at *different* velocities. For velocities near the speed of sound such that (12) is strongly violated, we see no dressed state formation. In those cases the field energy keeps growing via repeated emissions of grey solitons from the potential as observed in Ref. [10].

Finally, we want to point out the stability of our solution. Since the initial condition used in Fig. 2b is far away from the dressed state solution, the convergence of the field precisely into the analytical form (8) and (9) is an evidence of dynamic stability of dressed states. We have performed direct stability tests based on collective excitation frequencies in the co-moving frame. Our numerical results confirm the dynamic stability of the solutions [13]. The issue of the sensitivity to the shape of  $U$  has also been examined. By replacing  $U$  with a gaussian, we repeat the numerical calculations as in Fig. 2b. As long as the gaussian’s width and height are well within the domain described by inequalities (11) and (12), we see the emergence of dressed states. Therefore dressed state formation is not sensitive to the details of  $U$ .

To conclude, we have presented a family of motional dressed states as exact analytic solutions of a nonlinear Schrödinger field interacting with a moving potential. The formation of a dressed state enforces frictionless motion, and so it preserves a key feature in superfluid in the supersonic regime. We stress that our method is not limited to the solution family (8-9) and potentials in purely repulsive forms. With our approach, one can construct wide varieties of solutions of the field and the

corresponding matching potentials, as long as the stability and physical constraint for the potential are allowed. This opens interesting possibilities of controlling local densities of a condensate dynamically. Our study is for one-dimensional systems; we speculate that similar dressing features appear in Bose condensates in quasi-1D configurations, such as condensates trapped in a long cigar shape or in a ring, within characteristic time scales. It remains an important open question whether dressing effects appear in two or higher dimensional systems. We hope to address this issue in the future.

## ACKNOWLEDGMENTS

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  - [12] For example, in the case of  $\alpha = 1/v$ ,  $\beta = 1$  and  $v \gg 1$ , the potential is approximately independent of  $v$ , i.e.,  $U(\tau) \approx \text{sech}^2(\tau)$ . In this case, the dressed field solution  $r(\tau) \approx 1 + \text{sech}^2(\tau)/2v^2$  exists at any  $v \gg 1$ .
  - [13] In the co-moving frame  $x' = x - vt$ ,  $t' = t$ , the collective excitation frequencies with respect to a stationary state  $\psi$  (i.e., a dressed state) are defined by the eigenvalues of the matrix,

$$M = \begin{pmatrix} D + 2|\psi|^2 & \psi^2 \\ -\psi^{*2} & -D^* - 2|\psi|^2 \end{pmatrix}$$

where  $D \equiv -\frac{1}{2}\partial_{x'}^2 + iv\partial_{x'}$ . The system is dynamically stable if all eigenvalues of  $M$  are real.

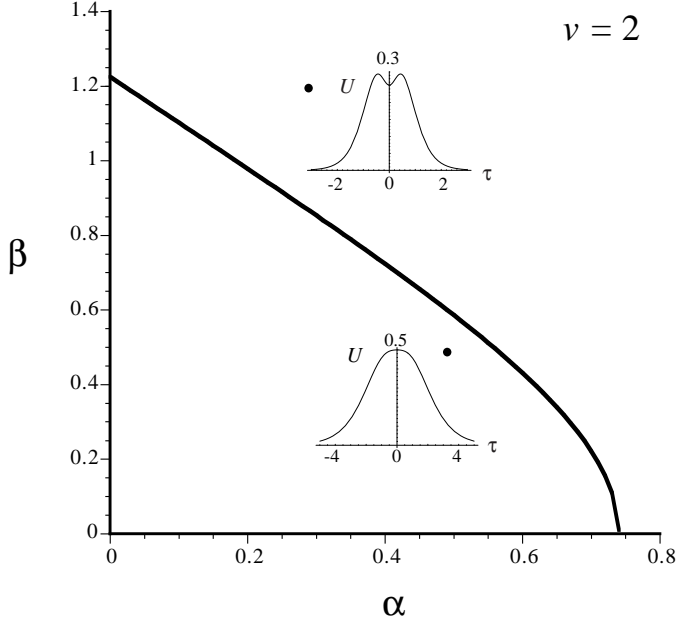


FIG. 1. Parameter space of the solutions at a supersonic speed  $v = 2$ , with positive  $\alpha$  and  $\beta$ . The characteristic curve is defined by the equality sign of (11). The region under the curve is associated with purely repulsive potentials  $U$ . The shapes of  $U$  at the two dots are representative examples above and below the curve

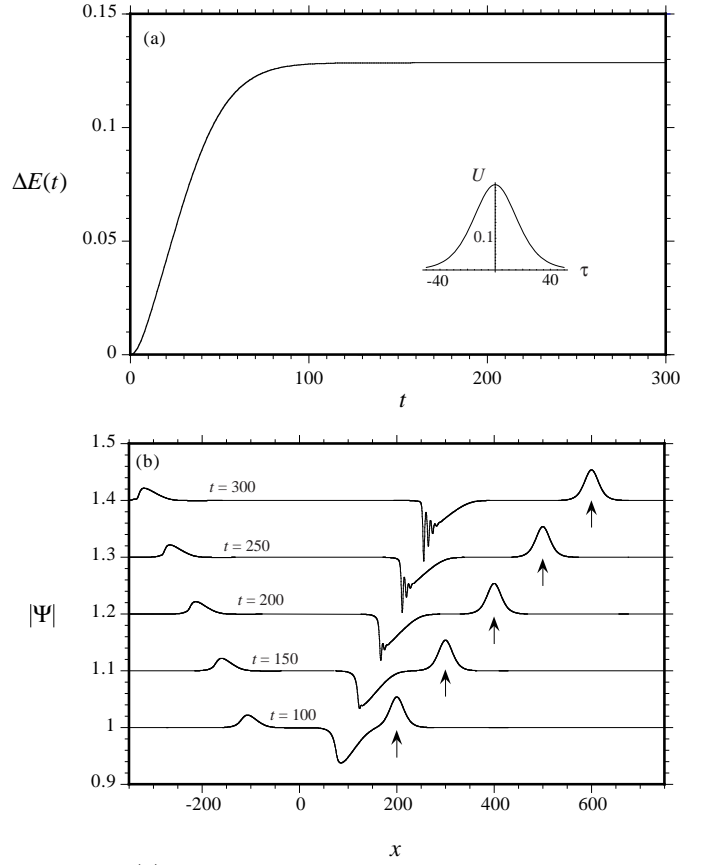


FIG. 2. (a) The increase of energy as a function of time with the parameters:  $v = 2$ ,  $\alpha = 0.2$  and  $\beta = 0.05$ . The potential is shown in the inset. (b) Time evolution of the modulus of  $\Psi$  at different times. The curves are shifted up by 0.1 successively in order to display the evolution. The arrows indicate the position of the center the potential at the corresponding time.